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DELAY DIFFERENTIAL EQUATION MODELS FOR REAL-TIME DYNAMIC SUBSTRUCTURING

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ABSTRACT

Real-time dynamic substructuring is a testing technique that models an entire system through the combination of an experimental test piece, representing part of the system, with a numerical model of the rest of the system. Delays can have a significant effect on the technique, as signals are passed between the two parts of the system in real-time. The focus of this paper is the influence of the delay on the dynamics of the substructured system. This is addressed using a linear example which may be described by a delay differential equation (DDE) model. This type of analysis allows critical delay values for system stability to be computed, which in turn can be used to help design the substructuring test system. Two methods are presented for the example considered. The first makes use of an analytical approach and the second of a numerical software tool, DDE-BIFTOOL. Normally, in substructuring tests, the actuators response time exceeds the critical delay time and the substructured system is unstable. It is demonstrated that the system can be stabilized using an adaptive delay compensation technique based on forward polynomial prediction. Finally we outline how these techniques may be applied to an industrial example of modelling a nonlinear spring.

INTRODUCTION

Real-time dynamic substructuring is a hybrid numerical-experimental testing technique that combines a *critical element*, tested experimentally, with a numerical model of the remainder of the system being considered. The combination of the two parts of the system during a real-time test is intended to mimic the behaviour of the complete or *emulated system*. The key challenge is to ensure that the combination of the critical element (or *substructure*) and the numerical model behave in the same way as the emulated system. So far hybrid testing has been developed successfully by using expanded time scales, known as pseudo-dynamic testing [1–4], which has the limitation that dynamic and hysteresis forces must be estimated. Implementing the process in real-time, eliminating the need for these estimations, is the subject of much recent research [5–7].

To carry out a dynamic substructuring test the substructure is identified and fixed into an experimental test rig. The interface interaction between the substructure and the numerical model is typically provided by electric or hydraulic actuators, which apply displacements on the substructure. The actuators act as a *transfer system* and are designed to follow the appropriate output displacements calculated by the numerical model [8]. To complete

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the coupling between the substructure and the numerical model the forces imposed by the substructure on the transfer system at the interface are measured and included within the numerical model. The whole testing process must take place in real-time to simulate the dynamic behaviours of the emulated system.

If the feedback forces are treated as an external influence (or forcing) on the numerically modelled part of the system, then the model can be described by a set of ordinary differential equations (ODEs). However, as it is not possible for any transfer system to react instantaneously, delay between the two parts of the model arise. Therefore, these forces are generally dependent on a system state subject to delay, and as a result the system is more accurately modelled with delay differential equations (DDEs) [9]. In addition to transfer system delay, other delays such as measurement, signal processing and computation delays are likely to occur. Typically, for mechanical systems these delays are large enough to have a significant influence on the overall dynamics of the substructured system.

In this paper the effect of delay errors that occur in a substructure experiment is considered using a mass-spring-damper example. For this system we show how the *critical delay* — the delay at which the substructured system loses stability — can be found using both an approximate analytical analysis and DDE-BIFTOOL.

Once the critical delay has been identified, a suitable control system can be developed to give a stable substructured system that reproduces the dynamics of the emulated system as closely as possible. We show results from an experimental implementation of the mass-spring-damper example, where the critical element is connected to a servo-mechanical actuator and the overall substructured system is stabilized by an adaptive delay compensation technique.

Single-step delay compensation techniques for real-time substructuring have been shown to improve accuracy [5, 10, 11]. In this paper a more generic approach to delay compensation based on polynomial forward prediction is used, which allows more flexibility in compensating for a range of signals. The experimental results show a high degree of correlation with the DDE modelling approach. Finally, we discuss a method of over-compensation, which can be used to help attain a successful real-time dynamic substructure test for a dynamical system with very low damping.

THE SPRING-MASS-DAMPER SYSTEM

The emulated system considered in this paper is the mass-spring-damper arrangement shown in Fig. 1. The equation of motion for the system can be written as,

$$m\ddot{z}^* + c(\dot{z}^* - \dot{r}) + k(z^* - r) = -k_s z^*, \quad (1)$$

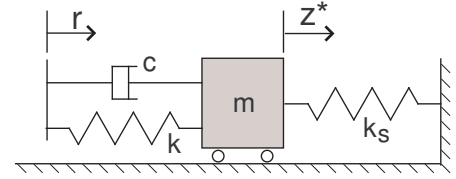


Figure 1. Schematic representation of the single mass-spring oscillator.

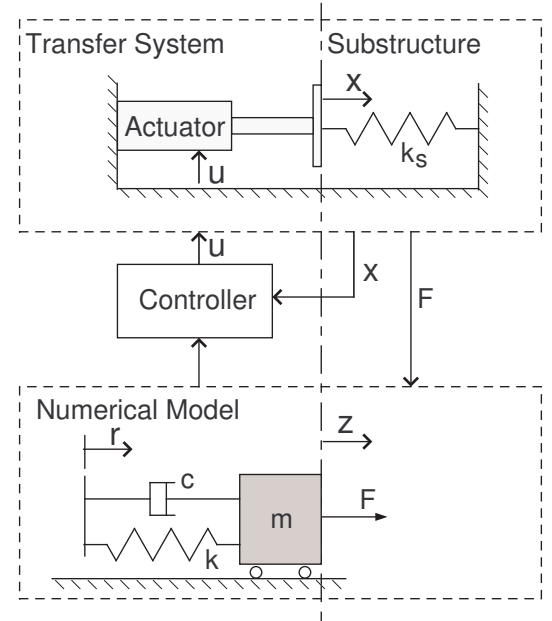


Figure 2. Schematic representation of a *substructured* system with one transfer system.

where, m, c, k and k_s are the mass, damping and two stiffness scalars, respectively, $r(t)$ is the support excitation and the state of the system is represented by z^* , where $(\cdot)^*$ is used to indicate the emulated system — more detailed analysis is given in [12].

To create the substructured model of the system shown in Fig. 1, the spring k_s is isolated and taken to be the critical element. The remainder of the structure, the excitation wall and the mass-spring-damper unit, is modelled numerically — as is shown schematically in Fig. 2.

The dynamics of the numerical model is

$$m\ddot{z} + c(\dot{z} - \dot{r}) + k(z - r) = F, \quad (2)$$

where the feedback force F is the substructure response of $F = -k_s x$. The transfer system cannot react instantaneously to the change of state of the numerical model which introduces a time delay such that $x(t) = z(t - \tau)$.

The delay τ introduces a systematic synchronization error $z(t) - x(t) = z(t) - z(t - \tau)$ into the substructuring algorithm. Therefore, when the synchronization error is non-zero, the force is described by the delayed state z of the numerical model $F = -k_s z(t - \tau)$. The overall substructured system is then governed by a delay differential equation (DDE) that can be written as

$$m\ddot{z} + c\dot{z} + kz + k_s z(t - \tau) = c\dot{r} + kr. \quad (3)$$

The substructured system is *unstable* if the synchronization error grows exponentially in time and *stable* if the synchronization error remains bounded.

1 STABILITY OF THE HYBRID SYSTEM

From Eqn. (3) for $r = \dot{r} = 0$ and with $x(t) = z(t - \tau)$ the complimentary equation is

$$m\ddot{z} + c\dot{z} + kz + k_s x = 0. \quad (4)$$

This can be expressed with non-dimensionalized parameters in the form

$$\frac{d^2 z}{d\hat{t}^2} + 2\zeta \frac{dz}{d\hat{t}} + z + px = 0, \quad (5)$$

where $\omega_n = \sqrt{\frac{k}{m}}$, $\hat{t} = \omega_n t$, $\hat{\tau} = \omega_n \tau$, $p = \frac{k_s}{k}$ and $\zeta = \frac{c}{2\sqrt{mk}}$.

The introduction of a delay term into a linear ordinary differential equation (ODE) changes the spectrum of the ODE by a perturbation of order τ and introduces infinitely many new modes. For a small delay the new modes are all strongly damped and the perturbation of the ODE spectrum can be expanded in the small parameter τ . The assumption that τ is small is reasonable for the parameters used in the experiments in this paper. First a search for solutions of the form $z = Ae^{\lambda \hat{t}}$ is performed. This leads to the characteristic equation for the system of

$$\lambda^2 + 2\zeta\lambda + 1 + pe^{-\lambda\hat{\tau}} = 0. \quad (6)$$

The complex roots λ_i of Eqn. (6) are the system eigenvalues; the sign of their real parts determines the stability of the system. For mechanical and structural systems, which are lightly damped, ζ is small and we can assume that $\hat{\tau}$ is small and expand $e^{-\lambda\hat{\tau}}$ to first order as $1 - \lambda\hat{\tau}$. Using this approximation, Eqn. (6) becomes

$$\lambda^2 + \lambda(2\zeta - p\hat{\tau}) + (1 + p) = 0. \quad (7)$$

Solving for λ gives the roots

$$\lambda_{1,2} = -\frac{1}{2}(2\zeta - p\hat{\tau}) \pm \frac{1}{2}\sqrt{(2\zeta - p\hat{\tau})^2 - 4(1 + p)}, \quad (8)$$

which govern the dominant eigenvalues for the DDE system given by Eqn. (3) when $\tau \ll 1$. For $\hat{\tau}$ small, we can make the assumption that the eigenvalues remain complex.

This means that $4(1 + p) > (2\zeta - p\hat{\tau})^2$ such that the real parts of the eigenvalues from Eqn. (8) determine the overall stability of the system, i.e. it is stable only if $p\hat{\tau} < 2\zeta$. This means that the system is stable if the delay τ is less than the critical delay

$$\tau_c = \frac{2\zeta}{p\omega_n} = \frac{c}{k_s}. \quad (9)$$

As the response delay, τ , increases past τ_c a Hopf bifurcation occurs [13–15]. It is also clear that for a lightly damped, high stiffness system τ_c will be small and, consequently, the control algorithm must work harder to maintain stability.

The second approach to determining the stability boundaries of Eqn. (5) is to search for points in the parameter space where the characteristic Eqn. (6) has purely imaginary solutions, that is, just undergoes a Hopf bifurcation [15]. The stability boundaries are found in the parameter space by searching for solutions of the form $z = Ae^{j\omega\hat{t}} = Ae^{j\hat{\omega}\hat{t}}$ where $\hat{\omega} = \frac{\omega}{\omega_n}$ is a positive real number (0 cannot be a characteristic root in this case). Substituting $\hat{\omega}$ into the characteristic equation gives

$$-\hat{\omega}^2 + 2\zeta\hat{\omega}j + 1 + pe^{-j\hat{\omega}\hat{\tau}} = 0. \quad (10)$$

Equating real and imaginary parts leads to

$$\hat{\tau} = \frac{1}{\hat{\omega}} \operatorname{arccot} \left(\frac{\hat{\omega}^2 - 1}{2\zeta\hat{\omega}} \right) + \frac{n\pi}{\hat{\omega}} \quad (11)$$

and

$$p = \sqrt{(\hat{\omega}^2 - 1)^2 + 4\zeta^2\hat{\omega}^2}. \quad (12)$$

Here n is an integer — if arccot is to be taken between 0 and π then n is non-negative (since $\hat{\tau}$ is positive).

In Fig. 3 the curves for $n = 0$ to $n = 3$ are shown in the $(\hat{\tau}, p)$ -plane with ζ fixed at 0.1066. They define the stability boundary; the grey area is the region of stability. The dashed curve in Fig. 3 is the stability boundary Eqn. (9) obtained from the perturbation analysis — clearly demonstrating that the approximation only holds for small values of the delay. The critical value $\hat{\tau}_c$ from

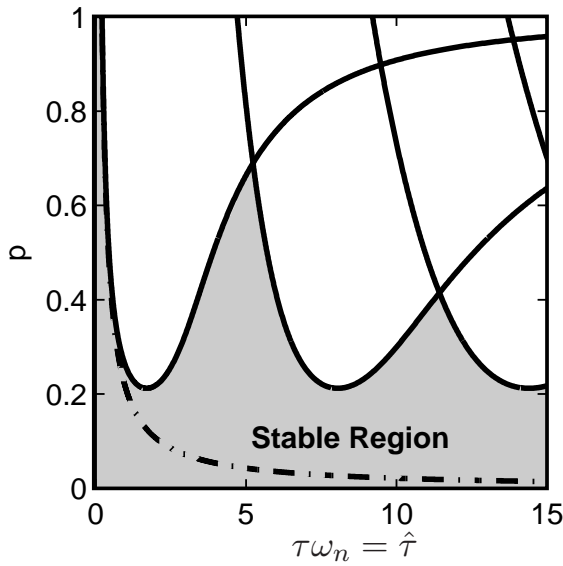


Figure 3. Non-dimensionalized complex root solutions of the DDE analysis for a range of response delays and substructure stiffnesses (solid line: exact solution, dashed line: perturbation solution.)

the perturbation analysis can be determined from Fig. 3 as $\hat{\tau}_c = 0.2132$, from which the critical time delay can be computed as $\tau_c = 6.67$ ms. This compares with $\hat{\tau}_c = 0.2165$ and $\tau_c = 6.77$ ms from the complex root solution.

1.1 Numerical Stability Analysis

For more complex systems it becomes impossible to find stability regions by using the analytical approach we have discussed above. In this case a numerical approach for finding the stability regions can be carried out using DDE-BIFTOOL. This tool is a collection of Matlab routines for numerical bifurcation analysis of systems of DDEs with multiple fixed, discrete delays; it is freely available for scientific purposes [16]. The package can be used to compute branches of steady state solutions (equilibria) and Hopf bifurcations. It also allows solutions to be continued, as system parameters are changed.

For the mass-spring-damper example the real parts of the roots of the characteristic equation, computed using DDE-BIFTOOL, are shown in Fig. 4 for the case where $p = 1$. The system loses stability when the first curve crosses the zero axis, corresponding to a pair of roots crossing the imaginary axis into the right half plane. This occurs at a value of $\tau_c = 6.77$ ms which agrees with the value found in the explicit stability analysis. This demonstrates how DDE-BIFTOOL can be used to model hybrid testing problems, especially those that are complex and/or non-linear systems.

2 EXPERIMENTAL SUBSTRUCTURE TESTING

2.1 Delay compensation

The technique discussed in this paper is an Adaptive Forward Prediction (AFP) algorithm, presented by Wallace *et al.* [17]. The AFP algorithm is a generic approach to delay compensation. It allows non-integer multiples of the previous time step to be predicted and adapts to changing plant conditions through self-tuning. Delay compensation is based on the idea of feeding a forward prediction z' of the numerical model state z of the into the transfer system. The AFP algorithm uses the prediction

$$z(t)' = (P_{N,n,\Delta}[z])(t + \rho) \quad (13)$$

where $P_{N,n,\Delta}[z]$ is the least-squares fitted N^{th} -order polynomial through the n time-point pairs $(t, z(t))$, $(t - \Delta, z(t - \Delta))$, ..., $(t - (n - 1)\Delta, z(t - (n - 1)\Delta))$. The time difference Δ is the sampling time step and ρ is the amount of forward prediction. Thus ρ is used to compensate for the delay τ generated by the control of the transfer system. The full AFP algorithm allows ρ to start from a set initial condition and includes an adaptive compensation for the amplitude inaccuracy. It is described in detail in Wallace *et al.* [17].

A fundamental difficulty for hybrid testing is that it is only safe to start an experimental test from a region of stability, — otherwise the system may destabilize if the controller cannot adapt to counteract for the delay effects quickly enough. Hence, it

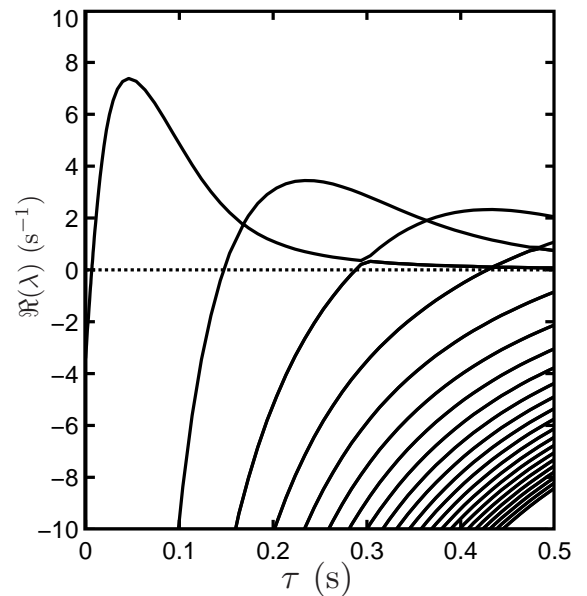


Figure 4. Real eigenvalue components of the characteristic equation for the stiffness parameter $p = 1$

is important for the performance of the AFP algorithm to find the interval of permissible ρ where the hybrid system with delay compensation, Eqn. (13), is stable. When the delay compensation of the AFP algorithm is applied the feed back force of the critical element changes to

$$F = -k_s(P_{N,n,\Delta}[z])(t + \rho - \tau). \quad (14)$$

System Eqns. (2) and (14) is now a DDE that depends on the values of z at the times $t - \tau, \dots, t - \tau - (n - 1)\Delta$. The interval of permissible ρ for the system parameters can be computed using DDE-BIFTOOL. Figure 5 shows how the real part of the dominant eigenvalues of Eqns. (2) and (14) varies with ρ . Figure 5(a) represents the stability of the AFP algorithm for a fitting polynomial $P_{N,n,\Delta}$ of order $N = 2$ for $n = 12$ previous values of z , and Fig. 5(b) corresponds to a polynomial of order $N = 3$ fitted to $n = 12$ previous values. Both prediction schemes are compared to the exact prediction (grey line) using

$$F = -k_s z(t + \rho - \tau) \quad (15)$$

for ρ within the interval from -20 ms to 45 ms. The vertical dashed lines indicate the parameter values $\rho = 0$ (short dashes, no forward prediction, all curves coincide here) and $\rho = 9.4$ ms (forward prediction equals the actual delay in the system, $\rho = \tau$). These calculations show how the stability of the substructured system with delay compensation Eqn. (14) depends on ρ . If the forward prediction could match $z(t - \tau + \rho)$ perfectly (grey line), all ρ greater than $\tau - \tau_c = 2.63$ ms would result in a stable system.

The polynomial forward prediction gives, in general, only a finite interval of stability for ρ , and for low order schemes the interval of permissible ρ is it is largest. In this example, stability ranges from $\rho \approx \tau - \tau_c = 2.63$ ms to $\rho_{\max} \approx 40$ ms for $N = 2$, $n = 12$ (as in Fig. 5(a)). Thus, for low N the AFP algorithm can start with an initial guess for ρ that is substantially larger than the delay τ . Increasing the order N of the fitting polynomial improves the accuracy of the prediction but, in general, shrinks the range of forward prediction ρ . For example, Fig. 5(b) shows that the maximal permissible ρ is at $\rho_{\max} \approx 23$ ms for $N = 3$, $n = 12$. Near ρ_{\max} another eigenvalue of system Eqns. (2) and (14) becomes dominant and unstable. Additionally, we note that the permissible order of N is limited by the noise that is fed back from the load transducer.

2.2 Experimental Results

Our implementation of real-time substructuring used a dSpace DS1104 R&D Controller Board running on hardware architecture of MPC8240 (PowerPC 603e core) at 250 MHz with 32 MB synchronous DRAM. The dSpace companion software ControlDesk was used for online analysis, providing soft

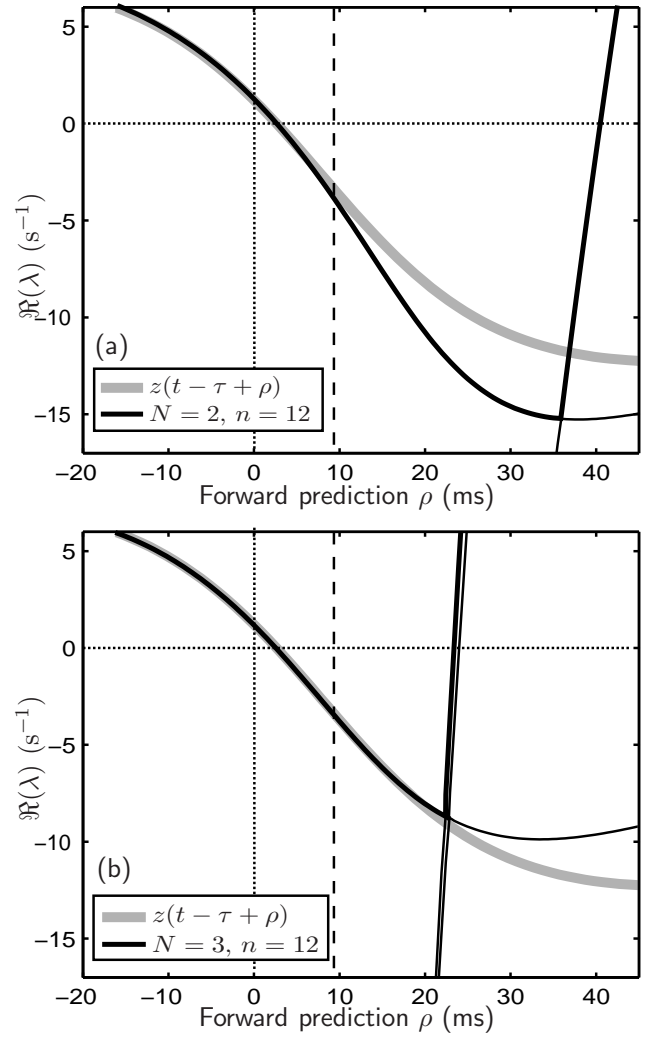


Figure 5. Real part of characteristic root eigenvalues for AFP strategy compared to the forward prediction using the exact value $z(t - \tau + \rho)$ for (a) $N = 2$ and (b) $N = 3$ with $n = 12$. Dominant eigenvalue is highlighted in bold; the long dashed line represents the transfer system delay for the experimental tests

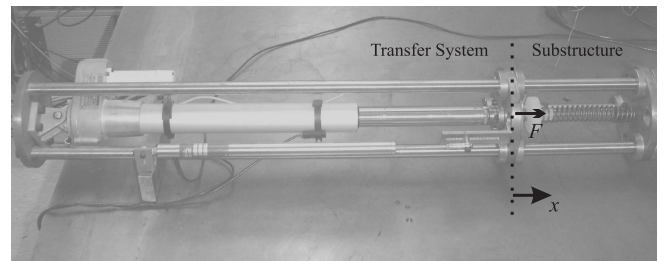


Figure 6. Experimental rig setup of substructured system.

real-time access to the hard real-time application. The system parameters were found through system identification, namely $m = 2.2\text{kg}$, $k = k_s = 2250\text{N/m}$ and $c = 15\text{Ns/m}$. The transfer system is a UBA (timing belt and ball screw configuration) linear Servomech actuator. Figure 6 shows the experimental rig setup of the substructured system.

Figure 7 shows typical experimental steady-state results for a wall excitation of 1.5Hz and *constant* delay compensation of $\rho = 9.4\text{ms}$ with polynomial fitting of $N = 3$ and $n = 12$. From Fig. 7(a) it is clear that the numerical model dynamics z closely replicate those of the emulated system z^* , losing significant accuracy only at direction changes of the actuator. The transfer system dynamics are shown via a synchronization subspace plot [17] in Fig. 7(b), where perfect synchronization is represented by a straight diagonal line. A constant delay turns this straight line into an ellipse, as can be seen from the limit of stability shown in grey, representing z vs. $z(t - \tau_c)$. The subspace plots show that there is generally a high level of synchronization, well below the stable limit, apart from when the actuators change direction.

The transition to instability can be seen from Fig. 8 for the experimental system which occurs when the forward prediction is reduced to $\rho = 2.6\text{ms}$. Since, in this example, the resulting delay of $\tau = 6.8\text{ms}$ is only slightly above the theoretical critical

delay of $\tau_c = 6.77\text{ms}$ the resulting exponential growth is relatively slow.

2.3 Over compensation

If delay is considered to be equivalent to adding negative damping in the system, then over compensating (predicting too far forward in time) will have the opposite effect of increasing the damping. Therefore, controlling the system to a *shifted* synchronization origin (we now take $\tau = -0.5\text{ms}$ as having zero synchronization error), will have the effect of over-damping the dynamic response of the numerical model. Firstly, this makes the numerical model slower to react to sudden state changes, i.e., high frequency noise fed back from the substructure. Secondly, it means that there is greater margin before the critical delay limit is reached, which constitutes a trade-off with the reduction in accuracy.

However, there is a more fundamental reason why it is significant to be able to operate the substructuring algorithm in an over-compensated region: to deal with substructuring tests when $\tau > \tau_c$. In this case (as with all substructuring tests) it is preferable to start the test using the measured force. However, this may lead to immediate instability and a failed test. A pragmatic approach that ensures stability is to initiate the test using a numerical estimation of the force (i.e., zero time delay) and switch over to the measured force when the control algorithm has achieved a high level of synchronization. Although the system is stable, now there is a corresponding loss of accuracy.

In the example considered here the actual response delay of $\tau = 9.4\text{ms}$ is greater than the critical delay of $\tau_c = 6.77\text{ms}$, meaning that the test is initiated in an unstable region. However, this may not necessarily lead to a catastrophic instability if the controller can respond faster than the unstable growth. When there is uncertainty about the critical element or transfer system(s) characteristics, the AFP algorithm should start with a low order N to give a large range of stable forward prediction ρ and to over compensate the initial guess. This gives the largest stable region as shown by Fig. 5. Once the adaptation algorithm is close to convergence, the prediction order can be increased to improve the accuracy of the substructuring experiment. The permissible order N is limited because the maximal stable ρ shrinks below τ for increasing N .

This can be seen in Fig. 9, where panel (a) shows the over compensation method, (b) the zero initial conditions method, and (c) the case of no delay compensation.

3 A PIECEWISE LINEAR EXAMPLE

The analysis of the spring-mass-damper example served to illustrate the application of the substructuring technique and the usefulness of DDE modelling. We now introduce an example of a genuinely nonlinear model that is directly motivated by a va-

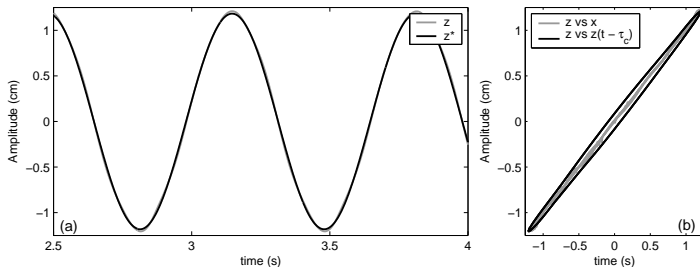


Figure 7. Experimental real-time substructure test with wall excitation of 1.5Hz and delay compensation of 9.4ms , (a) comparison of substructuring and emulated systems displacement and (b) synchronization subspace, with the limit of stability shown by the z vs $z(t - \tau_c)$ loop.

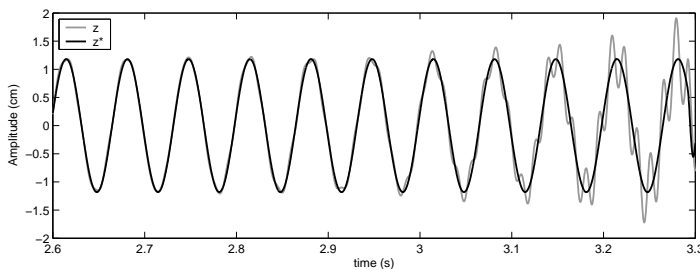


Figure 8. Transition to instability as the delay compensation is reduced on the experimental system; forward prediction of $\rho \approx 2.6\text{ms}$.

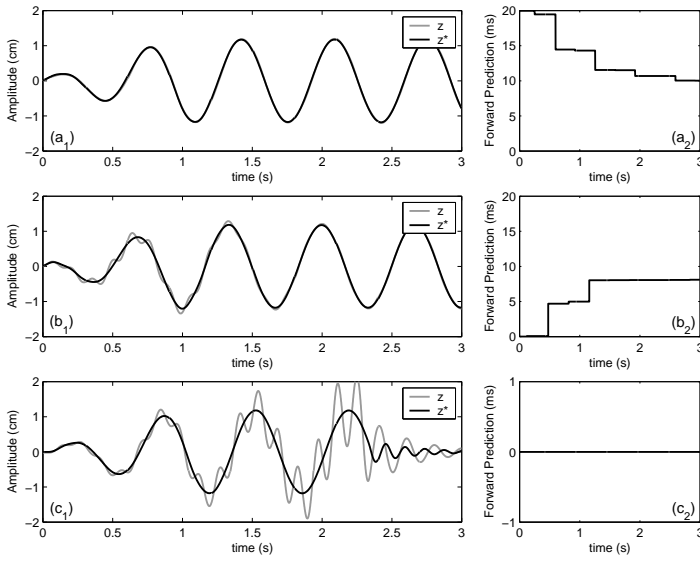


Figure 9. Experimental numerical model accuracy for differing control methodologies: (a) over compensation, (b) zero initial conditions, (c) no delay compensation (note this test was stopped after 2.3s due to excessive displacements). Controller adaption parameters: $\alpha = 75$, $\beta = 5$, $\gamma = 2$, $N = 3$, $n = 12$.

riety of engineering applications which include force saturation behaviour, such as occurs in the fields of mechatronics and actuation.

In these types of applications it is a valid approach to assume that the stiffness force is a nonlinear function that can be approximated by a piecewise linear stiffness function as is shown in Fig. 10.

We now have a nonlinear stiffness force, F_k , which is modeled by a piecewise linear function of the form

$$F_k(t) = \begin{cases} F_a & x \geq a \\ k_p x & -a < x < a \\ -F_a & x \leq -a. \end{cases} \quad (16)$$

In this case we have

$$m\ddot{z} + c\dot{z} + kz + F_k(t) = 0. \quad (17)$$

For $-a < x < a$, this can be expressed in the same way as Eqn. (5) as

$$\frac{d^2 z}{d\hat{t}^2} + 2\zeta \frac{dz}{d\hat{t}} + z + px = 0, \quad (18)$$

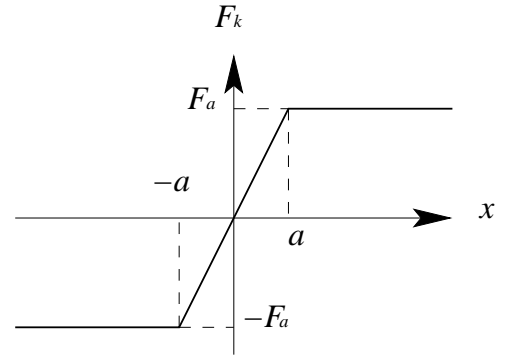


Figure 10. Piecewise linear stiffness model

where $\omega_n = \sqrt{\frac{k}{m}}$, $\hat{t} = \omega_n t$, $\hat{\tau} = \omega_n \tau$, $p = \frac{k_p}{k}$ and $\zeta = \frac{c}{2\sqrt{mk}}$. So that from the analysis in section 1 we get $\tau_c = \frac{c}{k_p}$. In this example, for $x \geq a$ and $x \leq -a$ there is no critical delay τ_c because F_a is constant and does not depend on the delayed state $z(t - \tau)$. This is simply another DDE to which the technique presented in this paper of using DDE-BIFTOOL may be applied. However, due to the piecewise linear nature of the problem, there may well be other modes of stability loss and complex behaviour which are as yet unexplored in the context of substructure testing. The analysis of effects of such nonlinearities constitutes an important future direction towards applying this work to real engineering systems.

4 CONCLUSION

In this paper the hybrid numerical-experimental technique of real-time dynamic substructuring has been considered. The example of a single mass-spring oscillator system has been used to demonstrate the effect of delay on the accuracy and stability of the substructuring algorithm. The substructured system has been represented as a delay differential equation (DDE) model and critical delay values were determined via analysis and with the software DDE-BIFTOOL.

In real-time dynamic substructuring, the delays arise through the control of the transfer systems, which are often larger than the critical delay. An example of an adaptive delay compensation scheme has been presented, which was validated by experimental measurements. Again with DDE-BIFTOOL we discussed an over compensation method for substructuring that can be used to initiate the test in a stable region of the substructuring algorithm. Finally, we briefly discussed the application of the DDE modelling to a nonlinear stiffness function that can be approximated by a piecewise linear function.

In future work the DDE approach will be used in more complex substructuring scenarios. The overall goal is to implement real-time dynamic substructuring of real engineering com-

ponents, such as cables of suspension bridges and sloshing tanks for high-rise buildings, and damper units for helicopter rotors.

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